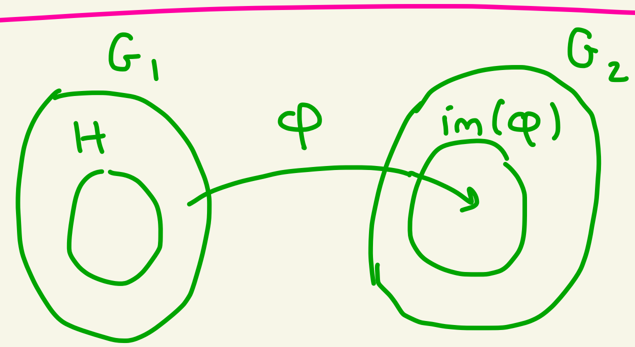
Math 4550

Topic 7 – First Isomorphism Theorem

Theorem (First Isomorphism Theorem)

Let $\varphi: G_1 \to G_2$ be a homomorphism between two groups G_1 and G_2.

Then,
$$G_1 / \ker(\varphi) \cong \operatorname{im}(\varphi)$$



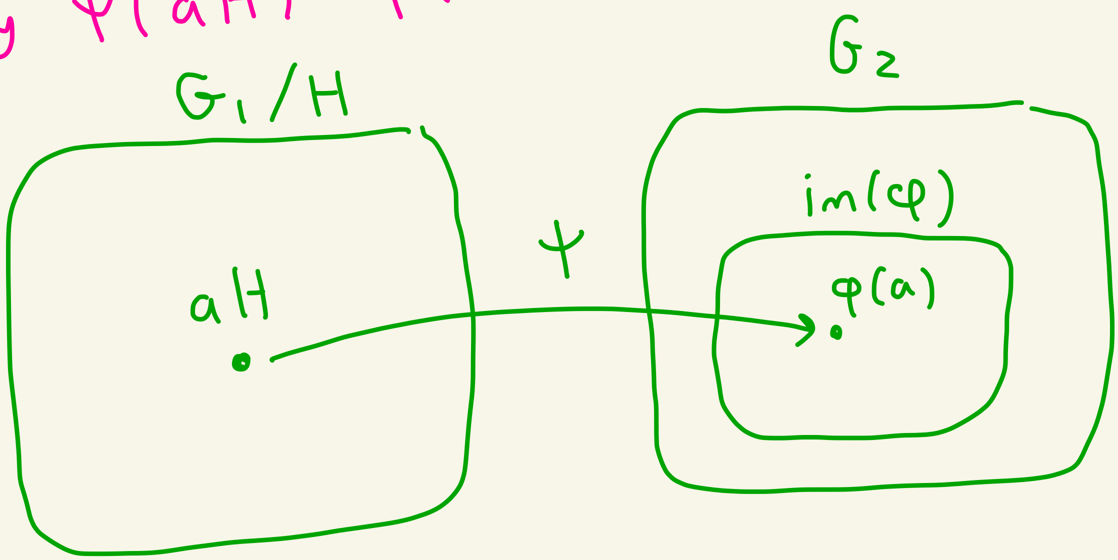
<u>proof:</u> Let $H = \ker(\varphi)$.

Then $H \trianglelefteq G_1$.

So, G_1/H is a group.

Define $\psi : G_1/H \to \operatorname{im}(\varphi)$

by $\psi(aH) = \varphi(a)$.



Claim 1: ψ is well-defined.

pf: Suppose $aH = bH$ for some $a, b \in G_1$.

Then, $b^{-1}a \in H$.

Since $H = Ker(\varphi)$ we have $\varphi(b^{-1}a) = e_2$
where e_2 is the identity of G_2.

So, $\varphi(b)^{-1}\varphi(a) = e_2$

Thus, $\varphi(a) = \varphi(b)$.

Hence, $\psi(aH) = \varphi(a) = \varphi(b) = \psi(bH)$.

So, ψ is well-defined. $\boxed{\text{claim 1}}$

Claim 2: ψ is an isomorphism

pf: Let $a, b \in G_1$.

Then

$$\psi((aH)(bH)) = \psi(abH) = \varphi(ab) \overset{\text{since } \varphi \text{ is a homomorphism}}{=} \varphi(a)\varphi(b) = \psi(aH)\psi(bH)$$

So, ψ is a homomorphism.

Let's show that ψ is one-to-one.

Suppose $\psi(aH) = \psi(bH)$.

Then $\varphi(a) = \varphi(b)$.

So, $\varphi(b)^{-1}\varphi(a) = e_2$.

Thus, $\varphi(b^{-1}a) = e_2$.

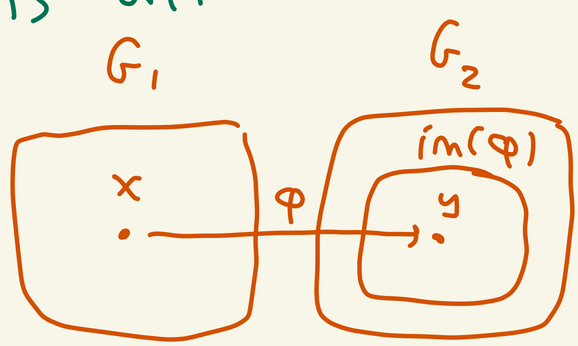
So, $b^{-1}a \in \ker(\varphi)$.

Thus, $b^{-1}a \in H$.

Hence, $aH = bH$.

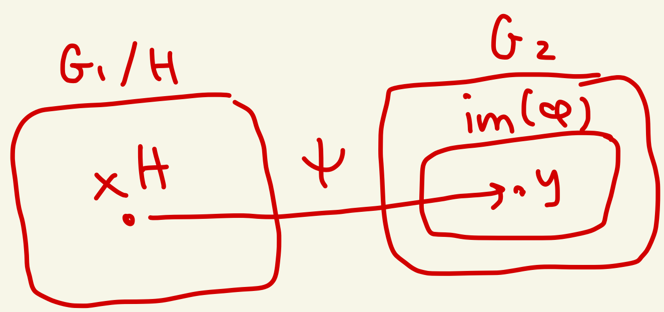
Therefore ψ is one-to-one.

Let's show that ψ is onto.



Let $y \in \operatorname{im}(\varphi)$.

Then there exists $x \in G_1$ with $\varphi(x) = y$.



Hence $xH \in G_1/H$ and $\psi(xH) = \varphi(x) = y$.

Thus, ψ is onto.

$\boxed{\text{Claim 2}}$

By claim 1 and claim 2 the theorem is proven.

Ex: Let $\varphi : GL(2,\mathbb{R}) \to \mathbb{R}^*$ be defined by $\varphi(A) = \det(A)$.

Recall that φ is a homomorphism with $H = \ker(\varphi) = SL(2,\mathbb{R})$.

Thus, $GL(2,\mathbb{R})/H = GL(2,\mathbb{R})/SL(2,\mathbb{R}) \cong \mathbb{R}^*$

Under the isomorphism $\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} H\right) = ad - bc$

$\boxed{\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} H\right) = \varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$